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COMPUTATION OF 2D POTENTIAL FIELDS: A PROPOSED APPROACH FOR IMPROVED VISUALIZATION

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Abstract **The capability of personal computers now-a-days is having a great line of difference with compared to the one we use to have in early sixties. This has a great impact on its computing utility. The present work throws light on the nature possibilities to explore them in a collective way for education and research for visualization.**

Keywords **Computation, Electric**

Introduction

The solution of numerical problems in theoretical physics was until a few years ago the domain of large computers. In recent years, personal computers have reached the power at par with large computers of early sixties. Apart from their high computational performances, personal computers offer interactive capabilities and rapid graphical output of results, which were not available even in the computers twenty years ago. Thus, personal computers offer us a wide field of possibilities in education and research. The present work is an attempt to visualize some of the two dimensional potential fields using personal computer thus opening a way to visualization problems through a theoretical approach.

Computation of 2D Electric Field-An approach to practical problem

Electrostatics and Magnetostatics are usually presented in a way Maxwell's equations are derived. With the aid of example one is introduced to their form of solution. The examples include from plate condenser to field created by current carrying conductor. The homogenous fields can be imagined, however picturing the penetration factor of the control grid of electronic valve or CRT, field causing acceleration and focusing in CRT takes strain. The present work throws light on the portion where the imagination succumbs to mere mathematical understanding.

In this section we shall deal with the electric field in vacuum pertaining to the regions and boundaries where Laplace's equation

$$\nabla^2 V=0 \qquad (1)$$

is obeyed followed by some typical examples for hands on experience and their interesting results in the

picture form along with the well-known equations.

(I) Formulation of the problem 2D Cartesian Coordinates

The boundary conditions we specify for the region in which the equation is to be solved.

$V = 0$, for $x = 0, y = a, y = 0$

and

$V = 100$ volts for $x = a$

The practical situation may be raised from the three conducting plates in the plane $V = 0$, for $x = 0, y = a, y = 0$ maintained at potential $V = 0$ and an isolated plate in the plane $x = a$ bearing $V = 100$ volts. Now one seeks potential function $V(x, y)$ in the region bounded by the plates $0 \le x, y \le a$.

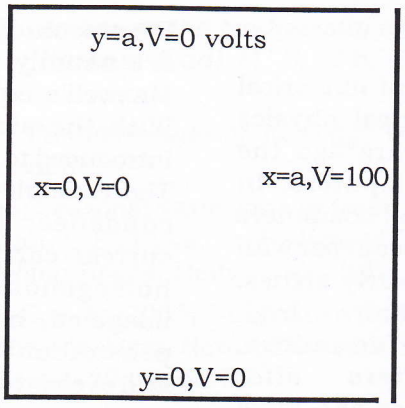


Fig.1 Geometrical Sketch of the accelerating anodes

The solution of Laplace's equation (1) is uniquely determined in the closed volume, when the potential is specified over surface of the volume. The volume in which we wish to compute the potential is still open.

(i) Discretisation of Laplace's equation

For numerical solution of Laplace's equation $\nabla^2 V(x, y) = 0$ we shall approximate $V(x, y)$ on the two dimensional grid of points with constant mesh width h. Instead of function $V(x, y)$ consider a matrix $(V_{i,k})$ of function values.

$$x_i = ih, \qquad y_k = kh$$
$$V_{ik} = V(x_i, y_k) \qquad (2)$$

Since the shape of electrodes chosen is simple, it makes easy to fit in the grid. Only we have to choose h so that the dimension of square a is integral multiple of h.

In order to solve the Laplace's equation (1) on this grid, we have to first discretise the differential operators $\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}$

Discretisation

$$\left.\frac{\partial V}{\partial x}\right|_{x+h} \doteq \frac{V_1 - V_o}{h}, \qquad \left.\frac{\partial V}{\partial x}\right|_{x-h} \doteq \frac{V_o - V_3}{h}$$

$$\frac{\partial^2 V}{\partial x^2} \doteq \frac{\left.\frac{\partial V}{\partial x}\right|_{x+h} - \left.\frac{\partial V}{\partial x}\right|_{x-h}}{h} = \frac{V_1 - V_o - V_o + V_3}{h^2}$$

Similarly, $\frac{\partial^2 V}{\partial y^2} \doteq \frac{\left.\frac{\partial V}{\partial y}\right|_{x+h} - \left.\frac{\partial V}{\partial y}\right|_{x-h}}{h} = \frac{V_2 - V_o - V_o + V_4}{h^2}$

Combining last two relations with (1), we get,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \doteq = \frac{V_1 + V_2 + V_3 + V_4 - 4V_o}{h^2} = 0$$

Thus $V_o \doteq \frac{1}{4}(V_1 + V_2 + V_3 + V_4)$ (3a)

In generalized notations $V_{i,j} \doteq \frac{1}{4}(V_{i,j-1} + V_{i,j+1} + V_{i+1,j} + V_{i-1,j})$ (3b)

(ii) Iteration

Employing a grid on the physical problem of dimension h and estimating the potential at central point using four specified boundary points as depicted in the Fig.2.

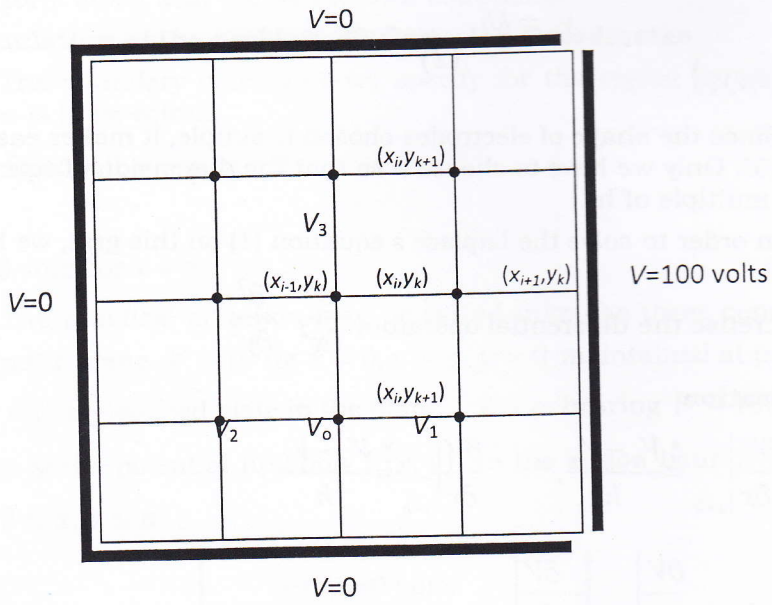


Fig.2 Calculation of potential by approximation formula

For simplicity consider a square region with conducting boundaries as shown in the Fig.1. The potential on right hand side conducting boundary is 100 volts and that of remaining sides zero. The problem is two dimensional and the sketch is cross section of the physical configuration. The region bounded by these four plates is divided into 16 squares, and some estimate of the potential be made at every corner before applying iterative method. Better the estimate, shorter the solution, although final result is independent of these initial estimates. Reasonably accurate values could be obtained from rough curvilinear-square map, or equation (3) could be applied to the large squares. At the center of the figure the potential estimate is then $\frac{1}{4}(V_1 + V_2 + V_3 + V_4) = \frac{1}{4}(100 + 0 + 0 + 0) = 25.0$ The potential may now be estimated at the centers of the four double sized squares by taking the average of the potentials at the four corners or applying (3) along the diagonal set of axis. For the right hand side double squares, we select the potential of 50V for the (the average of 100 and 0), and then $\frac{1}{4}(50 + 100 + 25 + 0) = 43.8$ and for the left hand side double square $\frac{1}{4}(0 + 25 + 0 + 0) = 6.2$ And so on.

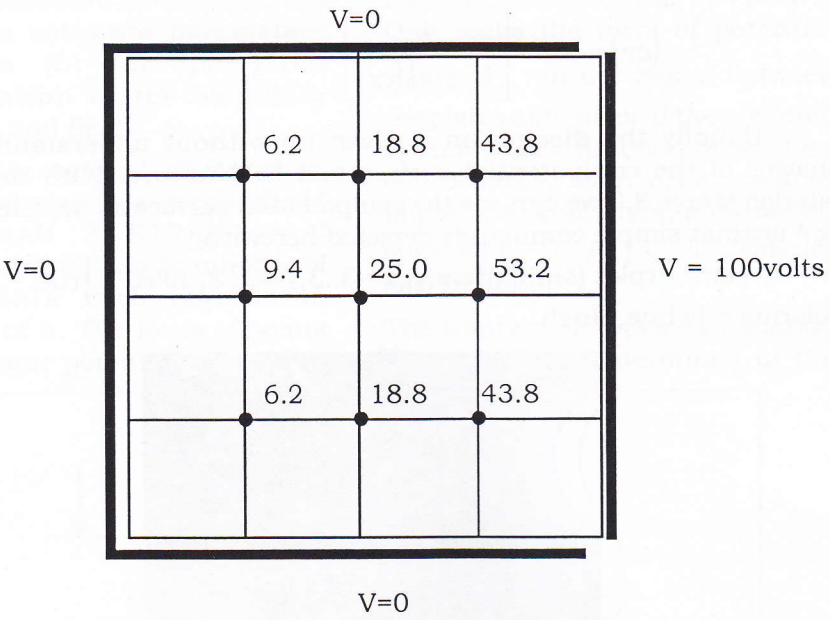


Fig.3 Calculation of potentials in 16 sub square corners

Mathematical Treatment-Solution to Differential Equation by Variable Separation

Let the functional dependence we seek be expressed as a product of two independent functions as $V(x,y) = V_1(x)V_2(y)$ Therefore, the Laplace's equation reduces to,

$$\frac{1}{V_1(x)}\frac{\partial^2 V_1(x)}{\partial x^2} = -\frac{1}{V_2(y)}\frac{\partial^2 V_2(y)}{\partial y^2} = c^2$$

$$V_1(x) = A\cosh(cx) + B\sinh(cx), \qquad V_2(y) = C\cos(cy) + D\sin(cy)$$

$$V = V_1(x)V_2(y) = \left[A\cosh(cx) + B\sinh(cx)\right]\left[C\cos(cy) + D\sin(cy)\right]$$

$$V = V_o \sinh(cx)\sin(cy), \qquad \text{Choose } V_o = BD,\ A = C = 0 \qquad (4)$$

$$V = 0 \text{ for } y = \frac{\pi}{c}, \frac{2\pi}{c}\ldots$$

$$V = 0 \text{ for } x = 0$$

The equipotential surface for k volts is

$$(cy) = \sin^{-1}\left[\frac{k}{V_o \sinh(cx)}\right] \tag{5}$$

Usually the discussion is given up without understanding the behavior of the equipotential surface and field nature. With the aid of Waterloo Maple 8.0 we can see the equipotential surface as depicted in the Fig.4 against simple commands depicted herewith.

>contourplot (sinh(x)*sin(y),x=-3..3,y=-3..3, filled = true, coloring = [white,blue])

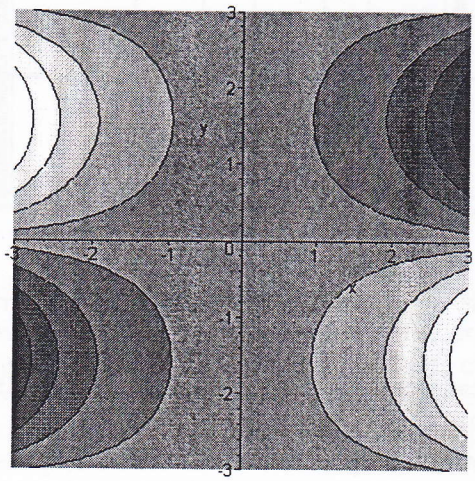


Fig.4. Equipotentials sketches of equation (5) in Maple 8.0

The knowledge of the potential is a key to other related important parameters/information such as:

Electric field strength using $\mathbf{E} = -\nabla V$

Density of flux $\mathbf{D} = \varepsilon \mathbf{E}$

Evaluate the density of flux on boundary surface $\mathbf{D} = \mathbf{D}_S = D_N \mathbf{a}_N$

Recognize surface density of charge $\rho_S = D_N$

Estimate total charge on the boundary surfaces $Q = \iint_{Surface} \rho_S dS$

The capacitance $C = \frac{|Q|}{V_o}$ (6)

Respective substitution will enable to estimate parameters listed in (6) for the given configuration of the boundary condition and field.

Thus, the method of iteration exercised manually or even with aid of a small FORTRAN or C programme will lead to number of data points with regards to selection of h. The locus of points holding same potential will lead to what we call equipotential and is resembling to what is achieved through Maple.

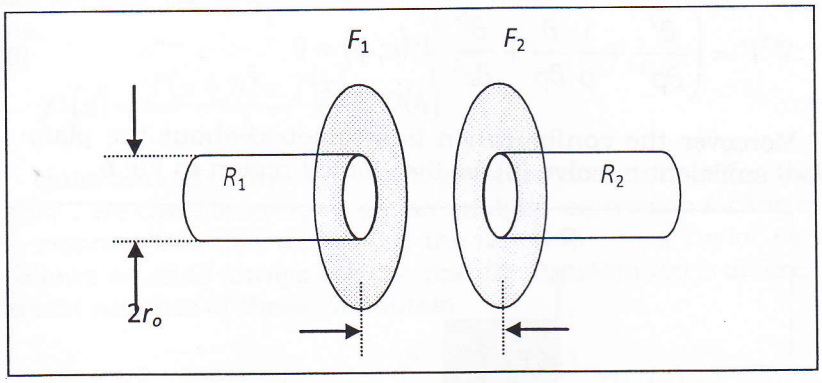


Fig. 5 Geometrical Sketch of the accelerating anodes

(II) Formulation of the problem 2D Cylindrical Coordinates

In this discussion we choose the boundary condition suitable to construct an electrostatic lens. As depicted in Fig. 5 it is a cylindrical metal tube with radius is interrupted by a flange .

The flanges form a plate condenser with two annular plates F1 and F2 at a separation of distance 2d to which two pieces of tube R1 and R2 are attached. Let the right hand flange be charged to a potential of 1000V and the left hand to the potential of -1000V. One seeks the form of potential $V(x, y, z)$ in the region between the plates and tubes. If the potential is known as a function of position coordinates, then other related important parameters/information such as E, D, ρ , Q, C using (6) can be found.

The solution of Laplace's equation (1) is uniquely determined in the closed volume, when the potential is specified over surface of the volume. The volume in which we wish to compute the potential is still open. Clearly to the far right and far left of the flanges we shall find $V_1 = +1000\text{V}$ and $V_2 = -V_1 = -1000\text{V}$, respectively, in the tubes. The potential field will change between flanges and in the far left and far right it will remain constant. Moreover, the potential field will have axial (or say Azimuthal) symmetry which will reduce the coordinates from 3 to 2 if we employ the Cylindrical Coordinate system for this

mathematical treatment. Let the axis of the tubes is z-axis. The coordinates x and y are replaced by ρ and φ where ρ denotes distance from the z-axis and φ azimuthal angle around z-axis. Thus the Laplace's equation in cylindrical coordinates takes form as:

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad (\forall\, \rho > 0) \qquad (7)$$

Employing the azimuthal symmetry condition, the potential field will be independent φ of , we get

$$\nabla^2 V = \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2}\right) V(\rho, \varphi) = 0 \qquad (8)$$

Moreover the configuration is symmetric about the plane . It is therefore sufficient to solve (8) for the shaded region in Fig.6.

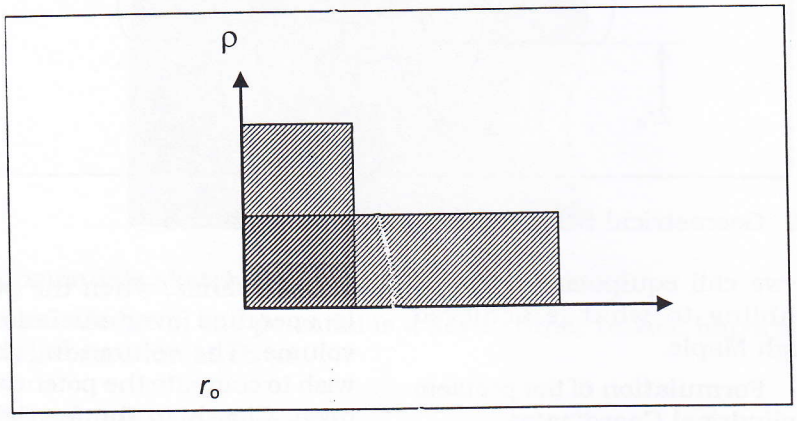


Fig.6 Shaded portion is symmetric about $z = 0$ and φ.

(i) Discretisation of Laplace's equation

For numerical solution of Laplace's equation we shall approximate $V(\rho, z)$ on the two dimensional grid of points with constant mesh width h. Instead of function $V(\rho, z)$ consider a matrix $(V_{i,k})$ of function values.

$$z_i = ih, \qquad \rho_k = kh$$
$$V_{ik} = V(z_i, \rho_k) \qquad (9)$$

Since the shape of electrodes chosen is simple, it makes easy to fit in the grid. Only we have to choose h so that the tube of radius r_o and the semi-distance of flange d are integral multiple of h.

In order to solve the Laplace's equation (9) on this grid, we have to first discretise the differential operators $\frac{\partial}{\partial \rho}, \frac{\partial^2}{\partial \rho^2}, \frac{\partial^2}{\partial z^2}$

Note: A simple approximation formula for numerical calculation of the first derivative of a function can be obtained from the definition of derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The transition to the limit $h \to 0$ can not be achieved on computer. Instead, we can use a small, but finite, value for h. Thus we obtain a formula.

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)$$

Unfortunately, this derivative gives us no estimate for the error term *O(h)* . We can however, obtain an analytic expression for the error in simple manner. For this we expand the function in a Taylor series; in what follows we shall always assume that the function *f(x)* is differentiable a sufficient number of times. We obtain

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \ldots$$

Substituting above equation in the last one we get

$$O(h) = -\frac{h}{2} f''(x) + \frac{h^2}{6} f'''(x) + \ldots$$

On computer one must not choose h too small, since below certain limit the rounding error becomes large than the result. One therefore seeks an approximation formula for which the errors go to zero with a high power of h. In the present this can be easily achieved. We subtract Taylor series for $f(x+h)$ the Taylor series for $f(x-h)$, and divide the difference by $2h$ to obtain approximation formula:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2) \qquad \text{(a)}$$

with the error term

$$O(h^2) = -\frac{h^2}{6} f'''(x) + \ldots$$

And similarly,

$$f''(x) = \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} + O(h^2) \quad \text{(b)}$$

with the error term

$$O(h^2) = -\frac{h^2}{6} f'''(x) + \ldots$$

(a) and (b) are known as two point formula for first and second derivative.

We have,

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O'(h^2)$$

$$f''(x) = \frac{f(x+h) - f(x-h) - 2f(x)}{h^2} + O''(h^2)$$

For $O'(h^2) = -\frac{h^2}{6} f^{(3)}(x) + \ldots, O''(h^2) = -\frac{h^2}{12} f^{(4)}(x) + \ldots$

Using above approximations

$$\nabla^2 V = \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) V(\rho, \varphi)$$

$$\nabla^2 V = \frac{1}{h^2} \left[V(\rho + h, z) + V(\rho - h, z) + V(\rho, z + h) + V(\rho, z - h) - 4V(\rho, z) \right]$$

$$+ \frac{1}{\rho} \frac{1}{2h} \left[V(\rho + h, z) + V(\rho - h, z) \right] + O(h^2) = 0$$

Applying relation (9) and neglecting error terms $O(h^2)$, multiply the equation by h2 and so obtain an approximation of Laplace's equation on grid for k>0

$$\left(V_{i,k+1} + V_{i,k-1} + V_{i+1,k} + V_{i-1,k} - 4V_{i,k} \right) + \frac{1}{2k} \left(V_{i,k+1} - V_{i,k-1} \right) = 0 \quad (10)$$

for $k = 0$ *i.e.* for the grid points on the axis of symmetry (10) is not valid. For these points we have to formulate a special equation. The equation is most simply obtained by first converting back to Cartesian coordinates x, y, z and using approximation formula.

$\nabla^2 V(x,y,z) =$

$$\frac{1}{h^2}\left[\begin{matrix} V(x+h,y,z)+V(x-h,y,z)+V(x,y+h,z)+V(x,y-h,z) \\ +V(x,y,z+h)+V(x,y,z-h)-6V(x,y,z)+O(h^2) \end{matrix}\right] = 0 \quad (11)$$

We wish to apply equation to the axis of symmetry and according to set $x = y = 0$ Now we return from potential function $V(x,y,z)$ in Cartesian coordinate system to the function $V(\rho,\phi)$ in Cylindrical coordinates. The first four function values on right hand side are all equal, namely equal to $V(h,z)$. If we neglect $O(h^2)$ and use relation (9) we obtain so called axis formula.

$$4V_{i,1} + V_{i+1,0} + V_{i-1,0} - 6V_{i,0} = 0 \quad (12)$$

The above equation (12) is analogous to (3) and may be exercised for iteration and graphical visualization manually (or through a small programme in FORTRAN or C).

ii) Method of Successive Over-Relaxation:

If one writes down (10) for all interior points of the grid and (12) for all the points on axis of symmetry, then one obtains a system of linear equations from which in principle the matrix (V_{ij}) can be calculated. Since we are to calculate many function values , however, this system of equations has a high dimension. The usual method for solving system of linear equations, such as the Gaussian elimination method, is no longer suitable for the solution of the problem.

The system of equations (10) and (12) however, has one peculiarity: each point is linked only with the directly neighboring points. If we represent Laplace's operator as an enormous matrix this matrix is almost empty: most of the elements are zero. For such a problem method of successive over-relaxation has been developed.

The system of linear equations is solved by iteration:

(i) One postulates an approximate solution.

(ii) From this one calculates by means of a formula a better approximate solution.

(iii) One test whether improved approximate solution fulfill a quality criterion.

For 1: For the known potential values in the interior region of the grid and on the axis of symmetry, one postulates the value which one feels to be reasonable. The method of successive over-relaxation converges even if one postulates a constant value for V by judicious choice one merely economies in iteration steps.

For 2: For each point (i,k) one after the other, in the interior and on the axis of symmetry, one applies (10) and (12) to calculate of a potential U at the point (i,k) from potential values at the neighboring points.

$$U = \begin{cases} \frac{1}{4}\left(V_{i,k+1} + V_{i,k-1} + V_{i+1,k} + V_{i-1,k}\right) + \frac{1}{8k}\left(V_{i,k+1} - V_{i,k-1}\right), & k > 0 \\ \frac{1}{6}\left(V_{i+1,k} + V_{i-1,k} + 4V_{i,k+1}\right), & k = 0 \end{cases} \quad (13)$$

The improved approximate solution at the point is calculated by means of formula:

$$V_{i,k}^{new} = V_{i,k}^{old} + \omega\left(U - V_{i,k}^{old}\right), \quad 1 \le \omega < 2 \quad (14)$$

The constant is chosen so that one obtains the fastest possible convergence. We shall now consider this in more detail. The $V_{i,k}$ are represented by merely one array of numbers in the memory. Whenever a new value of $V_{i,k}$ is calculated, the old value is over written by the new. The order in which (14) is applied to the point of grid is arbitrary.

For 3: When elements of the matrix of the potential values $\left(V_{i,k}\right)$ are no longer changing i.e.

$$V_{i,k}^{new} = V_{i,k}^{old} \quad (15)$$

then the matrix of the potential values $V_{i,k}$ satisfies the discretised Laplace's equation with the specified boundary condition. The quality or convergence criterion is that at no point (i,k) does $V_{i,k}^{new}$ by more than specified small value ξ In our example we shall specify $\xi = 10^{-7} V_1 = 10^{-4}\ V$

The peculiarity of the method is the parameter ω in (14). If one sets $\omega = 1$, then one simply replaces $\omega = 1$ by U. From (13) one can see

that U is calculated only from potential values at the neighboring grid points. At each application of (14), therefore, the information originating from the boundary values can only propagate by one mesh width. The propagation of information has a certain similarity to a diffusion process. With $\omega = 1$ the approximate values $V_{i,k}$ move slowly and monotonically in the course of iteration towards the respective end values. This process can be accelerated by setting $\omega > 1$ This is known as over-relaxation, in contrast to a relaxation with $\omega \leq 1$ If a too large value is chosen for ω then the oscillation and instability set in. It is known that, $\omega \geq 2$ always leads to instability in the method shown here. A suitable value for ω is obtained by trial. It is difficult to predict theoretically the most suitable value. However, it is known that too large value of ω converges rapidly becomes worse, so it is better to begin with smaller value of ω We shall replace the semi-separation distance of the flanges d and radius r_o by dimensionless integers m and n,

$$m = \frac{d}{h}, \qquad n = \frac{r_o}{h}$$

iii) Mathematical Treatment-Solution to Differential Equation by Variable Separation

Let the functional dependence we seek be expressed as a product of two independent functions as $V(\rho, z) = V_1(\rho)V_2(z)$ Therefore, the Laplace's equation reduces to,

$$\frac{1}{V_1(\rho)}\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\right)V_1(\rho) = -\frac{1}{V_2(z)}\frac{\partial^2 V_2(z)}{\partial z^2} = c^2 \tag{16}$$

$$V_1(\rho) = A_n J_n(c\rho) + B_n Y_n(c\rho), \quad V_2(y) = C\cos(cz) + D\sin(cz)$$

$$V = V_1(\rho)V_2(z) = \left[\alpha_o + \alpha_1\rho + \alpha_2\rho^2 + \alpha_3\rho^3 + \alpha_4\rho^4\right]\left[C\cos(cz) + D\sin(cz)\right]$$

$$V(\rho, z) = D\sin(cz)\left[\alpha_o + \alpha_1\rho + \alpha_2\rho^2 + \alpha_3\rho^3 + \alpha_4\rho^4\right]$$

Choose C = 0 (17)

The equipotentail surface for k volts will be:

$$D\sin(cz)\left[\alpha_o + \alpha_1\rho + \alpha_2\rho^2 + \alpha_3\rho^3 + \alpha_4\rho^4\right] = k \tag{18}$$

Note: Here the Bessel function contains a series of constants which are interdependent and expressed collectively as associated with even and odd powers in following manner.

$$\alpha_2 = \frac{c^2}{2^2}\alpha_0, \qquad \alpha_3 = \frac{c^2}{3^2}\alpha_1$$

$$\alpha_4 = \frac{c^2}{4^2}\alpha_2 = \frac{c^2c^2}{4^22^2}\alpha_0, \qquad \alpha_5 = \frac{c^2}{5^2}\alpha_3 = \frac{c^2c^2}{5^23^2}\alpha_1$$

$$\alpha_6 = \frac{c^2}{6^2}\alpha_4 = \frac{c^2c^2c^2}{6^24^22^2}\alpha_0, \qquad \alpha_7 = \frac{c^2}{7^2}\alpha_5 = \frac{c^2c^2c^2}{7^25^23^2}\alpha_1$$

(The above set of equations for α 's may be obtained if $V_1(\rho) = \sum_n \alpha_n \rho^n$

$\frac{\partial V_1}{\partial \rho} = \sum_n n\alpha_n \rho^{n-1}$ and $\frac{\partial^2 V_1}{\partial \rho^2} = \sum_n n(n-1)\alpha_n \rho^{n-2}$ in equation (16)).

With the aid of Waterloo Maple 8.0 we can see the equipotential surface as depicted in the Fig.7 against simple commands depicted. Here we have preferred all constants as unit for depicting nature equipotential surfaces.

The knowledge of the potential opens a key to find other related important parameters as mentioned in (6).

```
contourplot((1+(x^2+y^2)^(1/2)+(x^2+y^2)+(x^2+y^2)^(3/
2)+(x^2+y^2)^2)*sin(x),x=-3..3,y=-3..3,filled=true,
coloring=[white,blue]);

contourplot3d((1+(x^2+y^2)^(1/2)+(x^2+y^2)+(x^2+y^2)^(3/
2)+(x^2+y^2)^2)*sin(x),x=-3..3,y=-3..3,filled=true,
coloring=[red,blue]);
```

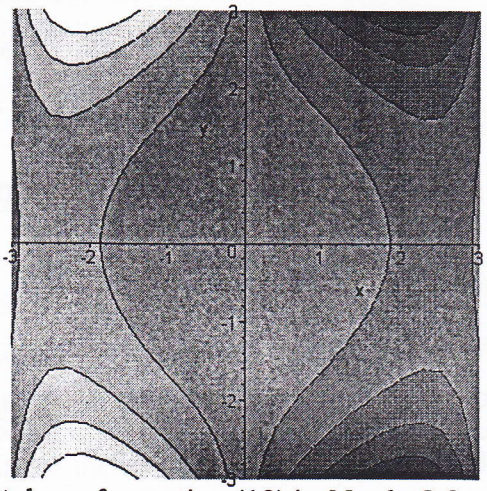


Fig 7. Equipotentials sketches of equation (18) in Maple 8.0

Families of Equipotential Surfaces

In this course of discussion of the above specific problem it would be very meaningful to look in to a very general approach to be exercised in the beginning while dealing non-routine problems-a test whether the family of *equipotentials* exits?.

If the potential function $V(x, y, z)$ is a solution of Laplace's, the one-parameter system of surfaces

$$V(x, y, z) = c \tag{20}$$

between functions V and f. Differentiating (19) and (20) partially with respect to x we obtain the result

$$\frac{\partial V}{\partial x} = \frac{\partial F}{\partial f}\frac{\partial f}{\partial x} \tag{21}$$

and the relation

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 F}{\partial f^2}\left(\frac{\partial f}{\partial x}\right)^2 + \frac{\partial F}{\partial f}\frac{\partial^2 f}{\partial x^2} \tag{22}$$

From which it follows that

$$\nabla^2 V = F''(f)(\nabla f)^2 + F'(f)\nabla^2 f \tag{23}$$

Now in free space, $\nabla^2 V = 0$ so that the required condition is that

$$\frac{\nabla^2 f}{(\nabla f)^2} = -\frac{F''(f)}{F'(f)} \tag{24}$$

Hence the condition that the surfaces (19) forms a family of equipotential surfaces in free space is that the quantity $\frac{\nabla^2 f}{(\nabla f)^2}$ is function of f alone.

If we denote this function by $\chi(f)$ then the equation (24) may be written as

$$\frac{d^2 F}{df^2} + \chi(f)\frac{dF}{df} = 0$$

From which it follows that

$$\frac{dF}{df} = A\exp\left(-\int \chi(f)df\right)$$

$$V = A\int exp\left(-\int \chi(f)df\right)df + B \qquad (25)$$

where A and B is a constant may be evaluated from the boundary condition.

Conclusion

With the little information that the potential is function of two variables we have started with the problem. Then we have fixed some boundary conditions in simple possible way and inferred many details of the field along with its visualization. The present generation of personal computers thus can be explored in visualization of fields and can emerge with highly interactive methods in education and research.

Appendix-1

Treatment to some differential equations in Maple 8.0

```
> PDE := 1/V1(x)*diff(V1(x),x,x)=-c^2;
```

$$PDE := \frac{\frac{d^2}{dx^2}\mathrm{V1}(x)}{\mathrm{V1}(x)} = -c^2$$

```
> ans := dsolve(PDE);
```

$$ans := \mathrm{V1}(x) = \_C1\sin(c\,x) + \_C2\cos(c\,x)$$

```
> PDE :=1/V2(y)*diff(V2(y),y,y)=c^2;
```

$$PDE := \frac{\frac{d^2}{dy^2}\mathrm{V2}(y)}{\mathrm{V2}(y)} = c^2$$

```
> ans := dsolve(PDE);
```

$$ans := \mathrm{V2}(y) = \_C1\,e^{(-c\,y)} + \_C2\,e^{(c\,y)}$$

```
>
> PDE := 1/V1(rho)*(diff(V1(rho),rho,rho)+1/rho*diff(V1(rho),rho))=c^2;
```

$$PDE := \frac{\left(\frac{d^2}{d\rho^2}\mathrm{V1}(\rho)\right) + \frac{\frac{d}{d\rho}\mathrm{V1}(\rho)}{\rho}}{\mathrm{V1}(\rho)} = c^2$$

```
> ans := dsolve(PDE);
```

$$ans := V1(\rho) = \_C1\ \mathrm{BesselI}(0, \rho\, c) + \_C2\ \mathrm{BesselK}(0, \rho\, c)$$

```
> PDE := 1/V2(phi)*diff(V2(phi),phi,phi)=-c^2;
>
```

$$PDE := \frac{\frac{d^2}{d\phi^2} V2(\phi)}{V2(\phi)} = -c^2$$

```
> ans := dsolve(PDE);
```

$$ans := V2(\phi) = \_C1\ \sin(c\,\phi) + \_C2\ \cos(c\,\phi)$$

```
> PDE := 1/V1(r)*(diff(V1(r),r,r)+2/r*diff(V1(r),r))=c^2;
```

$$PDE := \frac{\left(\frac{d^2}{dr^2} V1(r)\right) + \frac{2\left(\frac{d}{dr} V1(r)\right)}{r}}{V1(r)} = c^2$$

```
> ans := dsolve(PDE);
```

$$ans := V1(r) = \frac{\_C1\ \sinh(c\,r)}{r} + \frac{\_C2\ \cosh(c\,r)}{r}$$

```
> PDE := 1/V2(theta)*(1/r^2*diff(V2(theta),theta,theta)+cot(theta)/
r^2*diff(V2(theta),theta))=-c^2;
```

$$PDE := \frac{\frac{\frac{d^2}{d\theta^2} V2(\theta)}{r^2} + \frac{\cot(\theta)\left(\frac{d}{d\theta} V2(\theta)\right)}{r^2}}{V2(\theta)} = -c^2$$

```
> ans := dsolve(PDE);
```

$$ans := V2(\theta) = \frac{\_C1\ (\cos(\theta)-1)^{(1/4)}\ (\cos(\theta)+1)^{(1/4)}\ \mathrm{LegendreP}\left(\frac{\sqrt{1+4\,c^2\,r^2}}{2} - \frac{1}{2}, \cos(\theta)\right)}{\sqrt{\sin(\theta)}}$$

$$+\frac{\_C2\,(\cos(\theta)-1)^{(1/4)}\,(\cos(\theta)+1)^{(1/4)}\,\text{LegendreQ}\left(\frac{\sqrt{1+4\,c^2\,r^2}}{2}-\frac{1}{2},\cos(\theta)\right)}{\sqrt{\sin(\theta)}}$$

Appendix-2

Some calculations exercised in Maple 8.0

```
> with(VectorCalculus):
> Gradient( sinh(x)*sin(y), [x,y] );
```

$$\cosh(x)\sin(y)\,\bar{e}_x + \sinh(x)\cos(y)\,\bar{e}_y$$

```
> attributes( % );
```

$$vectorfield,\ coords = cartesian_{x,y}$$

```
> with(VectorCalculus):
> Gradient( (1+(x^2+y^2)^(1/2)+(x^2+y^2)^(2/2)+(x^2+y^2)^(3/2)+(x^2+y^2)^(4/2))*sin(z), [x,y,z] );
```

$$\left(\frac{x}{\sqrt{x^2+y^2}}+2\,x+3\sqrt{x^2+y^2}\,x+4\,(x^2+y^2)\,x\right)\sin(z)\,\bar{e}_x +$$

$$\left(\frac{y}{\sqrt{x^2+y^2}}+2\,y+3\sqrt{x^2+y^2}\,y+4\,(x^2+y^2)\,y\right)\sin(z)\,\bar{e}_y +$$

$$(\sqrt{x^2+y^2}+1+x^2+y^2+(x^2+y^2)^{(3/2)}+(x^2+y^2)^2)\cos(z)\,\bar{e}_z$$

```
> attributes( % );
```

$$vectorfield,\ coords = cartesian_{x,y,z}$$

```
>
```

$$vectorfield,\ coords = cartesian_{x,y}$$

```
> with(VectorCalculus):
> Gradient( (1+(x^2+y^2)^(1/2)+(x^2+y^2)^(2/2)+(x^2+y^2)^(3/2)+(x^2+y^2)^(4/2))*cos(z), [x,y,z] );
```

$$\left(\frac{x}{\sqrt{x^2+y^2}}+2x+3\sqrt{x^2+y^2}\,x+4\,(x^2+y^2)\,x\right)\cos(z)\,\bar{e}_x+$$

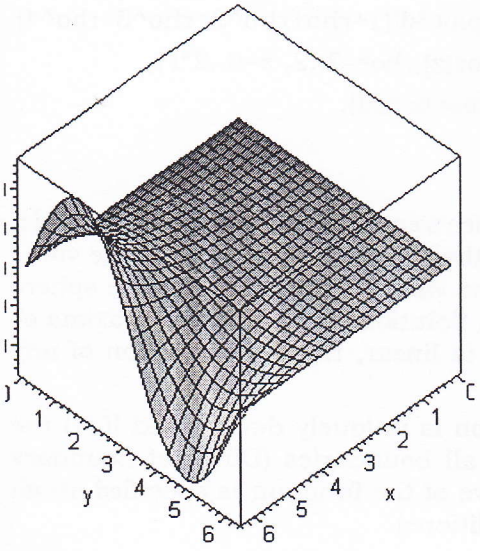
$$\left(\frac{y}{\sqrt{x^2+y^2}}+2y+3\sqrt{x^2+y^2}\,y+4\,(x^2+y^2)\,y\right)\cos(z)\,\bar{e}_y-$$

$$(\sqrt{x^2+y^2}+1+x^2+y^2+(x^2+y^2)^{(3/2)}+(x^2+y^2)^2)\sin(z)\,\bar{e}_z$$

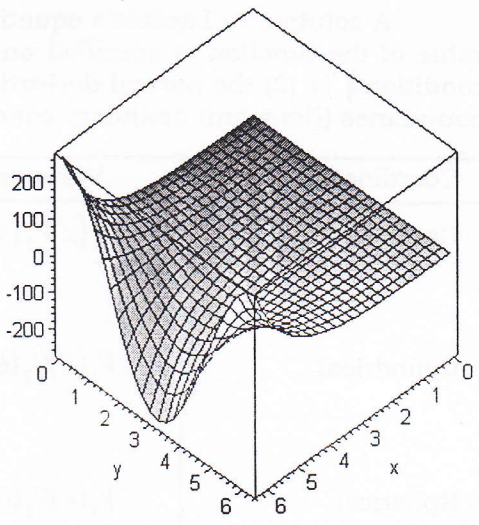
Appendix 3

Some visualizations of the functions and related functions.

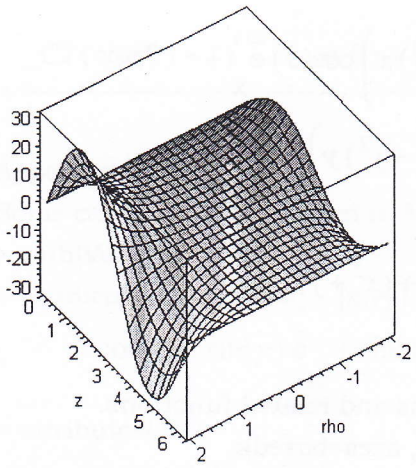
```
> plot3d(sinh(x)*sin(y),x=0..2*Pi, y=0..2*Pi, axes=boxed);
```



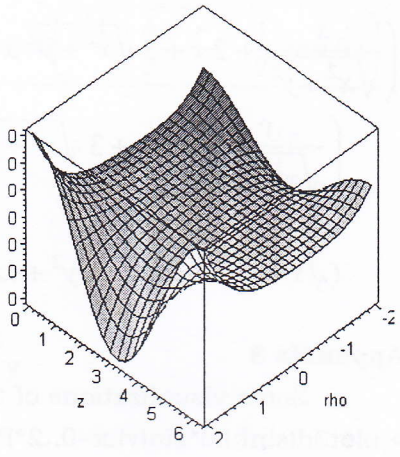

```
> plot3d(cosh(x)*cos(y),x=0..2*Pi,
y=0..2*Pi, axes=boxed);
```




```
> plot3d((1+rho+rho^2+rho^3+rho^4)*sin(z),rho=-2..2, z=0..2*Pi,
axes=boxed);
```



> plot3d((1+rho+rho^2+ rho^3+rho^4)*cos(z),rho=-2..2, z=0..2*Pi, axes=boxed);



> plot3d((1+rho+rho^2+rho^3+rho^4) *cos(z),rho=-2..2, z=0..2*Pi, axes=boxed);

Appendix 4

About Laplaces' Equation

A function which satisfies Laplace's equation is said to be harmonic. A solution to Laplace's equation has the property that the average value over a spherical surface is equal to the value at the center of the sphere (Gauss's harmonic function theorem). Solutions have no local maxima or minima. Because Laplace's equation is linear, the superposition of any two solutions is also a solution.

A solution to Laplace's equation is uniquely determined if (1) the value of the function is specified on all boundaries (Dirichlet boundary conditions) or (2) the normal derivative of the function is specified on all boundaries (Neumann boundary conditions).

Coordinate System	Variables	Solution Functions
Cartesian	$V_1(x)V_2(y)V_3(z)$	Exponential functions, circular functions, hyperbolic functions
Cylindrical	$V_1(\rho)V_2(\phi)V_3(z)$	Bessels functions, exponential functions, circular functions
Spherical	$V_1(r)V_2(\theta)V_3(\phi)$	Legendre polynomial, power, circular functions

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