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ORIGINAL ARTICLE

Five-dimensional cosmological model with a time-dependent equation of state in Wesson's theory

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Abstract Exact solution for a homogeneous cosmological model in 5D space-time-mass gravity theory proposed by Wesson (Astron. Astrophys. 119:145, 1983) is obtained by assuming the time-dependent equation of state. The behavior of the solution is discussed for the two cases $k < 0$ and $k = 0$. It is found that the observed constancy of the rest mass of an isolated particle in the present era may be interpreted as a consequence of the decreasing rate of change of rest mass with time. Moreover, a spontaneous compactification-like phenomenon of an extra dimension takes place in the case of $k = 0$. It is also found that with decrease in extra space the observable three-dimensional space entropy increases, thus accounting for the large value of entropy observable at present.

Keywords Cosmology · Wesson's theory · Variable rest mass · Time dependant equation of state

1 Introduction

Wesson (1983) proposed a 5D space-time-mass gravity theory in which the rest mass of a typical particle may change with time. Wesson introduced the fifth coordinate $x^4 = (\frac{G}{c^2})m$ (c is velocity of light and m is the rest mass) besides the 4D space-time coordinates and extended Einstein's general relativity from the 4D space-time to the 5D space-time-mass directly. It is useful to find and investigate solutions of the field equations in the 5D space-time-mass gravity theory to understand the meaning of the fifth dimensional subspace and provide predictions, which can be used to test the theory itself. Several authors, have, recently obtained exact solutions in Wesson's theory with or without matter distribution (Chatterjee 1986, 1987; Fukui 1987; Grøn 1988; Chatterjee et al. 1990; Banerjee et al. 1990b; Berman and Som 1993). But exact solutions with time-dependent equation of state are not much known in the literature. Haji and Boutros (1991) obtained solutions for an LRS Bianchi-I model with time dependant equation of state while Manna and Bhui (1994) presented higher dimensional cosmological model with a time dependent equation of state. Recently Bhui et al. (2005) has generalized the work of Haji and Boutros (1991) using Kaluza-Klein metric.

Following the work of Bhui et al. (2005) we find, in this paper, exact solutions of the field equations of Wesson (1983) space-time-mass gravity theory with a time dependent equation of state $p = \alpha(t)\rho$. Physical behaviors of the solutions are also discussed.

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2 Field equations

Following Grøn (1988) we take the line element in the form

$$ds^2 = dt^2 - \frac{R^2}{1 + \frac{kr^2}{4}}(dx^2 + dy^2 + dz^2) - A^2 dm^2, \tag{1}$$

where k characterizes the spatial curvature. Unlike Wesson, the fifth coordinate is taken to be space-like and the metric coefficients are assumed to be functions of time only.

We consider Einstein's equations

$$R_{ij} - \frac{1}{2}g_{ij}R = T_{ij} \quad (i, j = 0, 1, 2, 3, 4), \tag{2}$$

where the energy-momentum tensor T_{ij} for a perfect fluid is as suggested by Grøn (1988)

$$T_{ij} = \mathrm{diag}(\rho, -p, -p, -p, -p_4). \tag{3}$$

Here ρ is the matter density, p is isotropic pressure and p_4 is the pressure that would result if the fluid existed in a five dimensional space.

In the case of comoving fluid the field equations (2) with energy momentum tensor (3) lead to the equations

$$\frac{\dot{R}^2 + k}{R^2} + \frac{\dot{R}\dot{A}}{RA} = \frac{1}{3}\rho, \tag{4}$$

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2 + k}{R^2} + \frac{2\dot{R}\dot{A}}{RA} + \frac{\ddot{A}}{A} = -p, \tag{5}$$

$$\frac{\ddot{R}}{R} + \frac{\dot{R}^2 + k}{R^2} = -\frac{1}{3}p_4, \tag{6}$$

where dot $(\cdot)$ denotes derivative with respect to time and p_4 is the pressure that would result if the fluid existed in a five dimensional space time as it is possible in the Kaluza-Klein models. However according to Wesson's theory the fifth dimension is essentially a parametrisation of rest mass and we must have $p_4 = 0$. Equation (6) then reduces to

$$\frac{\ddot{R}}{R} + \frac{\dot{R}^2 + k}{R^2} = 0, \tag{7}$$

which gives

$$R^2 = K_1 + K_2 t - kt^2, \tag{8}$$

where K_1 and K_2 are arbitrary integrating constants.

Following Grøn (1988) one can calculate from geodesic considerations the variation of rest mass with time as

$$\frac{dm}{dt} = \frac{P_m}{A^2}\left[K^2 - \frac{P_m^2}{A^2}\right]^{-\frac{1}{2}}, \tag{9}$$

where P_m (a constant of motion) is the conjugate momentum and K is an arbitrary constant.

3 Solutions of field equations

3.1 Case 1 ($k < 0$)

In this case, one can adjust K_1 and K_2 such that (8) becomes a perfect square, we get

$$R = t. \tag{10}$$

Assuming the time dependent equation of state

$$p = \alpha(t)\rho. \tag{11}$$

From (4) and (5) we get a differential equation of the form

$$\frac{2\ddot{R}}{R} + \left(\frac{\dot{R}^2 + k}{R^2}\right)(3\alpha + 1) + \frac{\dot{R}\dot{A}}{RA}(3\alpha + 2) + \frac{\ddot{A}}{A} = 0. \tag{12}$$

From (10) and (12) we obtain

$$t^2\ddot{A} + t(3\alpha + 2)\dot{A} + (3\alpha + 1)(k + 1)A = 0. \tag{13}$$

Using the condition of exactness for the above linear equation and after a straightforward calculation we get

$$\alpha(t) = c_1 t^k - \left(\frac{k + 1}{3k}\right), \tag{14}$$

where c_1 is an arbitrary constant of integration.

Putting this value of $\alpha(t)$ in (13) we get

$$t^2\ddot{A} + \left[3c_1 t^{k+1} + \left(\frac{k - 1}{k}\right)t\right]\dot{A} + \left[3c_1 t^k - \frac{1}{k}\right]A = 0, \tag{15}$$

whose first integral is given by

$$t^2\dot{A} + \left[3c_1 t^{k+1} - \left(\frac{k + 1}{k}\right)t\right]A = c_2, \tag{16}$$

where c_2 is an arbitrary integrating constants.

From (7) and (10) we get $k = -1$. Using this in (16) and solving for A we get

$$A = \frac{c_2}{3c_1} + c_3 e^{3c_1/t}, \tag{17}$$

where c_3 is an arbitrary integrating constant.

This solution (17) is similar to the solution obtained by Bhui et al. (2005).

Now we consider $c_2 \cong -c_2$ then A can be put in the form

$$A = c_3 e^{3c_1/t} - \frac{c_2}{3c_1}. \tag{18}$$

Using (18) in the field equations it also follows that

$$\rho = \frac{9c_1}{t^3[\frac{c_2}{3c_1c_3}e^{-3c_1/t} - 1]}, \tag{19}$$

and

$$p = \alpha(t)\rho = \frac{9c_1^2}{t^4[\frac{c_2}{3c_1c_3}e^{-3c_1/t} - 1]}. \tag{20}$$

For physical validity i.e. for matter density to be nonnegative, one has to choose integration constants c_1, c_2 and c_3 to be positive and

$$\frac{c_2}{3c_1c_3} > 1, \tag{21}$$

here c_1 is related to the mass density of the system. When $c_1 = 0$, the mass density vanishes. We may be assured that the matter density is nonnegative. It follows from equation (19) that the integration constants c_1 and c_2 will always be positive. We discuss the dynamical behaviors of the model. The four volume $V = R^3 A$ starts from zero at $t = 0$ and vanishes at

$$t_0 = \frac{3c_1}{\ln[\frac{c_2}{3c_1c_3}]}. \tag{22}$$

It is interesting to note that the extra scale factors starting from an infinite expansion at the big bang reduces to Planckian length at the same point of time t_0. It has been conjectured (Sahdev 1984) that certain quantum gravity effects stabilized the extra space at the Planckian length and the cosmology of the universe evolves according to FRW model with the space remaining the silent spectator from this point on.

Using (18) in (9) we get

$$\frac{dm}{dt} = \frac{P_m}{[c_3 e^{\frac{3c_1}{t}} - \frac{c_2}{2c_1}][K^2(c_2 e^{\frac{3c_1}{t}} - \frac{c_2}{3c_1})^2 - P_m^2]^{\frac{1}{2}}}. \tag{23}$$

As $t \to \infty$, $\frac{dm}{dt}$ approaches to a constant value proportional to P_m provided $K(c_3 - \frac{c_2}{3c_1}) > P_m$.

3.2 Case 2 ($k = 0$)

In this case, for $K_1 = 0$, $K_2 \neq 0$ one can calculate from (8), the scale factor $R(t)$ as

$$R = \sqrt{K_2 t}. \tag{24}$$

Using (24) in (12), we get linear differential equation of the form

$$4t^2\ddot{A} + 2t[2 + 3\alpha]\dot{A} + [3\alpha - 1]A = 0. \tag{25}$$

From the condition of exactness of the above linear differential equation one can calculate $\alpha(t)$ as

$$\alpha(t) = 1 + d_1 t^{-1/2}, \tag{26}$$

where d_1 is an arbitrary constant of integration.

Substituting the value of $\alpha(t)$ from (26) in (25), we get

$$4t^2\ddot{A} + 2t(5 + 3d_1 t^{-1/2})\dot{A} + (2 + 3d_1 t^{-1/2})A = 0, \tag{27}$$

whose first integral is given by

$$4t^2\dot{A} + [2t + 6d_1 t^{1/2}]A = d_2, \tag{28}$$

where d_2 is an arbitrary constant of integration.

Equation (28) gives

$$A = \frac{1}{t^{1/2}}\left[\frac{d_2}{6d_1} + d_3 e^{3d_1/t^{1/2}}\right], \tag{29}$$

where d_3 is an arbitrary constant of integration.

Now we consider $d_2 \cong -d_2$ then (29) can be put in the form

$$A = \frac{1}{t^{1/2}}\left[d_3 e^{3d_1/t^{1/2}} - \frac{d_2}{6d_1}\right]. \tag{30}$$

As $t \to \infty$, $A \to 0$. Hence the solution is amenable to dimensional reduction.

Now using (30) in the field equations we get

$$\rho = \frac{9d_1}{4t^{5/2}[\frac{d_2}{6d_1d_3}e^{-3d_1/t^{1/2}} - 1]}, \tag{31}$$

and

$$p = \alpha(t)\rho = \frac{9d_1}{4[\frac{d_2}{6d_1d_3}e^{-3d_1/t^{1/2}} - 1]}\left[\frac{1}{t^{5/2}} - \frac{1}{t^3}\right]. \tag{32}$$

To ensure the matter density to be nonnegative it follows from (31) that integration constants d_1, d_2 and d_3 will be always positive and

$$\frac{d_2}{6d_1d_3} > 1.$$

In this case our solution is the generalization of the solution obtained by Manna and Bhui (1994).

4 Conclusion

We have presented here the five dimensional world as defined according to the Wesson's theory of gravitation with an energy momentum tensor containing matter density ρ, isotropic pressure p and pressure p_4 that would result if the fluid existed in a five dimensional space. From the solution of the metric coefficient A, it follows that over time A approaches zero, giving rise to phenomenon of the dimension reduction for an expanding models. From the expression for the mass density and pressure, it is shown that they are in general positive and vanishes at infinity with an initial singularity at $t = 0$.

From the energy conservation relation we get from (3)

$$\dot{\rho} + 3(\rho + p)\frac{\dot{R}}{R} + \rho\frac{\dot{A}}{A} = 0. \tag{33}$$

Since $n \sim R^{-3}$ the effective four-dimensional specific entropy E will in general increase at a rate (Banerjee et al. 1990a)

$$E \approx \frac{1}{KT}\left[\frac{d(\rho/n)}{dt} + \frac{pd(1/n)}{dt}\right], \tag{34}$$

where

$$E = \dot{\sigma}_4$$

$$= \frac{1}{nKT}\left[\dot{\rho} + 3(\rho + p)\frac{\dot{R}}{R}\right], \tag{35}$$

which in view of (9) yields

$$E = \frac{1}{nKT}\left[-\frac{\rho\dot{A}}{A}\right]. \tag{36}$$

From (36), we thus arrive at a very important relation which needs further interpretation in higher-dimensional physics. It may be conjectured that besides the well-understood mechanism of entropy increases due to dissipative phenomenon like viscosity, here the process of dimensional reduction also produces entropy in the fourth dimension, which may be enormous if A decreases quickly.

From (25) one can calculate the variation of rest mass thoroughly discussed by Grøn (1988). For this reason we have not given the explicit expression for $\frac{dm}{dt}$ in our case. It also follows from (25) that, in an open universe $k = -1$, $\frac{dm}{dt}$ becomes constant for large value of t. If one takes the value of this constant as either zero or extremely small then observed constancy of rest mass of an isolated particle in the present era may then be interpreted as a consequences of decreasing rate of change of rest mass with time.

In case of closed universe $k = 0$, $A \to 0$ as $t \to \infty$, which gives the dimensional reduction phenomenon. In this scenario the length of the fifth dimensional subspace shrinks while the normal dimension expands. In this case one finds that the 5D universe of Wesson enters in the usual 4D universe after a long period of expansion with the matter content as radiation. Here the process of dimensional reduction also produces entropy in the higher dimension, which may be enormous if A decreases quickly.

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